# Tractability in Value-based Argumentation

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**Abstract.** Value-based argumentation frameworks (VAFs) have proven to be a useful development of Dung's seminal model of argumentation in providing a rational basis for distinguishing mutually incompatible yet individually acceptable sets of arguments. In classifying argument status within value-based frameworks two main decision problems arise: *subjective acceptance* (SBA) and *objective acceptance* (OBA). These problems have proven to be somewhat resistant to efficient algorithmic approaches (the general cases being NP–complete and coNP–complete) even when very severe limitations are placed on the structure of the supporting Dung-style framework. Although using the number of *values* ($k$) represented within a given VAF leads to fixed parameter tractable (FPT) methods, these are not entirely satisfactory: the rate of growth of the parameter function ($k!$) making such methods unacceptable in cases where $k$ is moderately large, e.g. $k \geq 20$. In this paper we consider an alternative approach to the development of practical algorithms in value-based argumentation. In particular cases this leads to polynomial (in $|\mathcal{X}|$)methods, i.e. *irrespective of the value of* $k$. More general examples are shown to be decidable in $O(f(k)|\mathcal{X}|^2)$ steps where $f(k) = o(k!)$ resulting in worst-case run times that significantly improve upon enumerating all value orderings.

**Keywords.** Computational properties of argumentation; value-based argumentation frameworks; subjective and objective acceptance;

## Introduction

The standard argumentation framework (AF) approach of Dung [8] models argumentation via a directed graph, $\langle \mathcal{X}, \mathcal{A} \rangle$, wherein $\mathcal{X}$ is a finite set of atomic *arguments* and $\mathcal{A} \subseteq \mathcal{X} \times \mathcal{X}$ defines the *attack* relation over these: thus $\langle x, y \rangle \in \mathcal{A}$ (read as "$x$ attacks $y$") provides an abstract representation of the property that the arguments $x$ and $y$ are incompatible. Dung's model has been augmented to the concept of *value-based* argumentation frameworks (VAF) by Bench-Capon [2] so that the structure $\langle \mathcal{X}, \mathcal{A} \rangle$ becomes $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$. Here $\mathcal{V}$ is a finite set of $k$ abstract *values*[1] and $\eta : \mathcal{X} \to \mathcal{V}$ associates each argument with the value underlying it.

Although VAFs provide a number of powerful semantic benefits, as discussed in [3], [4], there are non-trivial computational problems. In particular, the fact

[1]The notion of "value" is qualitative – describing, e.g, ethical, social, political, etc. values – rather than quantitative.

that the two important decision questions in VAFs – subjective (SBA) and objective (OBA) acceptance – concern properties of *orderings* of $\mathcal{V}$, as opposed to properties of *subsets* of $\mathcal{X}$ has been shown to raise significant algorithmic issues. While restricting the structure of $\langle \mathcal{X}, \mathcal{A} \rangle$ by various means is known to lead to efficient methods for all of the semantics proposed within Dung's model, similar restrictions have proven less effective within VAFs. Thus it is known that requiring $\langle \mathcal{X}, \mathcal{A} \rangle$ to be *acyclic* or *symmetric* or *bipartite* suffices to yield polynomial time decision methods as shown in Dung [8], Coste-Marquis *et al.* [6], and Dunne [9]. In contrast even if $\langle \mathcal{X}, \mathcal{A} \rangle$ is a binary tree (a subset of those frameworks that are both bipartite and acyclic) no reduction in complexity results [9].[2]

Of course within the framework of *fixed parameter tractable* (FPT) methods, [7], it would appear that VAF computations are efficiently dealt with: problems are computable in $O(k!|\mathcal{X}|)$ steps so that both SBA and OBA are fixed parameter tractable with respect to the parameter $k = |\mathcal{V}|$. Such an approach – enumerate all possible value orderings, testing each in turn for the property of interest – fails, however, to be entirely satisfactory. This is not (solely) on account of the $k!$ growth rate for the parameter function - many feasible FPT methods involve significantly faster growing functions – but rather because the parameter itself ($|\mathcal{V}|$) may, in many cases, be moderately large. Ideally, FPT methods exploit parameters whose value is small in typical instances. In contrast there are natural settings of VAFs in which $|\mathcal{V}| \geq 15$ rendering algorithms with $15!|\mathcal{X}|$ steps unreasonable.

Our aim in this paper is to consider the following questions.

A. Are there classes of VAF, in addition to systems in which $\langle \mathcal{X}, \mathcal{A} \rangle$ is symmetric, for which SBA and OBA are polynomial time decidable (in $|\mathcal{X}|$) irrespective of $|\mathcal{V}|$?
B. To what extent can the $k!$ term be improved (possibly by limiting the structure of $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$) to yield FPT methods with run-time $O(f(k)|\mathcal{X}|^r)$ (where $r$ is some small constant) and $f(k)$ is significantly smaller than $k!$?

We obtain positive answers to both questions. In particular we describe a general category of restricted forms for $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$ and algorithms on these having run-time $O(k|\mathcal{X}|^2)$ for both of the principal decision questions. We further describe approaches guaranteeing worst-case run-time of $O(2^{ck} \times |\mathcal{X}|^2)$, where $c \leq 1$ is constant. Although $2^{ck}$ still imposes unrealistic requirements for very large numbers of values, it compares favourably with, and significantly improves upon, $k!$ (which is asymptotically $2^{O(k \log k)}$).

An important feature of the conditions leading to improved methods is that these combine structural restrictions on $\langle \mathcal{X}, \mathcal{A} \rangle$ *together with* restrictions on the *mapping* $\eta : \mathcal{X} \rightarrow \mathcal{V}$. In other words, the class of systems are not defined using purely graph-theoretic forms of the type analysed in [9].

We review background concepts in Section 1 and in Section 2 describe the main results of this paper, introducing the concept of *value graphs* in Sect. 2.1 and

[2]Symmetric frameworks do result in efficient VAF methods, however, these are rather more a consequence of a natural consistency assumption placed on $\eta$. We note that the behaviour commented upon arises in other augmentations of Dung's approach. In particular the so-called weighted systems of Dunne *et al.* [10] not only fail to yield polynomial time decision methods when $\langle \mathcal{X}, \mathcal{A} \rangle$ is acyclic but also fail to do so when $\langle \mathcal{X}, \mathcal{A} \rangle$ is symmetric.

showing how, under certain conditions, these lead to polynomial time methods in Section 2.2. The properties of value graphs and the cases considered in Section 2.2 motivate consideration of the extent to which such improvements may result for more general classes of value graph. We consider natural developments of this type in Section 3. Conclusions are given in Section 4.

## 1. Preliminaries: AFs and VAFs

The following concepts were introduced in Dung [8].

**Definition 1** *An* argumentation framework *(*AF*) is a pair* $\mathcal{H} = \langle \mathcal{X}, \mathcal{A} \rangle$*, in which* $\mathcal{X}$ *is a finite set of* arguments *and* $\mathcal{A} \subseteq \mathcal{X} \times \mathcal{X}$ *is the* attack relationship *for* $\mathcal{H}$*. A pair* $\langle x, y \rangle \in \mathcal{A}$ *is referred to as '*$y$ is attacked by $x$*' or '*$x$ attacks $y$*'. The convention of excluding "self-attacking" arguments is assumed, i.e. for all* $x \in \mathcal{X}$, $\langle x, x \rangle \notin \mathcal{A}$*. For* $R$*,* $S$ *subsets of arguments in the* AF $\mathcal{H}(\mathcal{X}, \mathcal{A})$*, we say that* $s \in S$ *is* attacked *by* $R$ *– written* $attacks(R, s)$ *– if there is some* $r \in R$ *such that* $\langle r, s \rangle \in \mathcal{A}$*. For subsets* $R$ *and* $S$ *of* $\mathcal{X}$ *we write* $attacks(R, S)$ *if there is some* $s \in S$ *for which* $attacks(R, s)$ *holds;* $x \in \mathcal{X}$ *is* acceptable with respect to $S$ *if for every* $y \in \mathcal{X}$ *that attacks* $x$ *there is some* $z \in S$ *that attacks* $y$*;* $S$ *is* conflict-free *if no argument in* $S$ *is attacked by any other argument in* $S$*.*

*A conflict-free set* $S$ *is* admissible *if every* $y \in S$ *is acceptable w.r.t* $S$*;* $S$ *is a* preferred extension *if it is a maximal (with respect to* $\subseteq$*) admissible set;* $S$ *is a* stable extension *if* $S$ *is conflict free and every* $y \notin S$ *is attacked by* $S$*;*

*For* $S \subseteq \mathcal{X}$*,*

$$\begin{array}{lll} S^- & =_{\text{def}} & \{\, p \;:\; \exists\, q \in S \;\; \textit{such that } \langle p, q \rangle \in \mathcal{A} \} \\ S^+ & =_{\text{def}} & \{\, p \;:\; \exists\, q \in S \;\; \textit{such that } \langle q, p \rangle \in \mathcal{A} \} \end{array}$$

*An argument* $x$ *is* credulously accepted *if there is* some *preferred extension containing it;* $x$ *is* sceptically accepted *if it is a member of* every *preferred extension.*

Bench-Capon [2] develops the concept of "attack" from Dung's model to take account of values.

**Definition 2** *A* value-based argumentation framework *(*VAF*), is defined by a triple* $\mathcal{H}^{(\mathcal{V})} = \langle \mathcal{H}(\mathcal{X}, \mathcal{A}), \mathcal{V}, \eta \rangle$*, where* $\mathcal{H}(\mathcal{X}, \mathcal{A})$ *is an* AF*,* $\mathcal{V} = \{v_1, v_2, \ldots, v_k\}$ *a set of* $k$ values*, and* $\eta : \mathcal{X} \to \mathcal{V}$ *a mapping that associates a value* $\eta(x) \in \mathcal{V}$ *with each argument* $x \in \mathcal{X}$*.*

*An* audience *for a* VAF $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$*, is a binary relation* $\mathcal{R} \subset \mathcal{V} \times \mathcal{V}$ *whose (irreflexive) transitive closure,* $\mathcal{R}^*$*, is asymmetric, i.e. at most one of* $\langle v, v' \rangle$*,* $\langle v', v \rangle$ *are members of* $\mathcal{R}^*$ *for any distinct* $v$*,* $v' \in \mathcal{V}$*. We say that* $v_i$ is preferred to $v_j$ in the audience $\mathcal{R}$*, denoted* $v_i \succ_{\mathcal{R}} v_j$*, if* $\langle v_i, v_j \rangle \in \mathcal{R}^*$*. We say that* $\alpha$ *is a* specific *audience if* $\alpha$ *yields a* total *ordering of* $\mathcal{V}$*. The notation* $\mathcal{U}$ *is used for the set of all specific audiences over* $\mathcal{V}$

A standard assumption from [2] which we retain in our subsequent development is the following:

*Multivalued Cycles Assumption* (MCA)
For any *simple cycle* of arguments in a VAF, $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$, – i.e. a finite sequence of arguments $y_1 y_2 \ldots y_i y_{i+1} \ldots y_r$ with $y_1 = y_r$, $|\{y_1, \ldots, y_{r-1}\}| = r - 1$, and $\langle y_j, y_{j+1} \rangle \in \mathcal{A}$ for each $1 \leq j < r$ – there are arguments $y_i$ and $y_j$ for which $\eta(y_i) \neq \eta(y_j)$.

In less formal terms, this assumption states every simple cycle in $\mathcal{H}^{(\mathcal{V})}$ uses at least two distinct values.

Using VAFs, ideas analogous to those introduced in Defn. 1 are given by relativising the concept of "attack" using that of *successful* attack with respect to an audience. Thus,

**Definition 3** *Let $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$ be a* VAF *and $\mathcal{R}$ an audience. For arguments $x$, $y$ in $\mathcal{X}$, $x$ is a* successful attack *on $y$ (or $x$* defeats *$y$)* with respect to the audience $\mathcal{R}$ *if:* $\langle x, y \rangle \in \mathcal{A}$ and *it is* not *the case that $\eta(y) \succ_{\mathcal{R}} \eta(x)$.*

Replacing "attack" by "successful attack w.r.t. the audience $\mathcal{R}$", in Defn. 1 yields definitions of "conflict-free", "admissible set" etc. relating to value-based systems, e.g. $S$ is conflict–free w.r.t. to the audience $\mathcal{R}$ if for each $x$, $y$ in $S$ it is not the case that $x$ successfully attacks $y$ w.r.t. $\mathcal{R}$. It may be noted that a conflict-free set in this sense is not necessarily a conflict-free set in the sense of Defn. 1: for $x$ and $y$ in $S$ we may have $\langle x, y \rangle \in \mathcal{A}$, provided that $\eta(y) \succ_{\mathcal{R}} \eta(x)$, i.e. the value promoted by $y$ is preferred to that promoted by $x$ for the audience $\mathcal{R}$.

Bench-Capon [2] proves that every specific audience, $\alpha$, induces a unique preferred extension within its underlying VAF: for a given VAF, $\mathcal{H}^{(\mathcal{V})}$, we use $P(\mathcal{H}^{(\mathcal{V})}, \alpha)$ to denote this extension: that $P(\mathcal{H}^{(\mathcal{V})}, \alpha)$ *is* unique and can be constructed efficiently, is an easy consequence of the following fact, implicit in [2].

**Fact 1** *For any* VAF, $\mathcal{H}^{(\mathcal{V})}(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle)$ *(satisfying* MCA*) and specific audience $\alpha$, the framework induced by including only attacks in the set*

$$\mathcal{B} \;=\; \mathcal{A} \setminus \{\langle x, y \rangle \;:\; \eta(y) \;\succ_\alpha\; \eta(x)\}$$

*is* acyclic.

**Proof:** Suppose the contrary and let $y_1 y_2 \ldots y_r$ (with $y_r = y_1$) be any simple cycle in the VAF $\langle \langle \mathcal{X}, \mathcal{B} \rangle, \mathcal{V}, \eta \rangle$ defined from $\mathcal{H}^{(\mathcal{V})}$ via the specific audience $\alpha$. Since each of the attacks $\langle y_i, y_{i+1} \rangle$ for $1 \leq i \leq r-1$ occurs in $\mathcal{A} \cap \mathcal{B}$ from the definition of $\mathcal{B}$ we must have

$$\forall\ 1 \leq i \leq r-1 \;\; \neg(\eta(y_{i+1}) \;\succ_\alpha\; \eta(y_i))$$

That is,

$$\forall\ 1 \leq i \leq r-1 \;\; (\eta(y_i) \;\succ_\alpha\; \eta(y_{i+1})) \bigvee (\eta(y_i) = \eta(y_{i+1}))$$

With some minor abuse of notation, we write $v \succeq_\alpha w$ if $(v = w) \;\vee\; v \succ_\alpha w$, so that the expression above implies

$$\eta(y_1) \succeq_\alpha \eta(y_2) \succeq_\alpha \ldots \succeq_\alpha \eta(y_{r-1}) \succeq_\alpha \eta(y_1) \succeq_\alpha \ldots$$

Since $\alpha$ is a specific audience so that $\succeq_\alpha$ is a total ordering, the only possible choice of values which this behaviour could arise is

$$\eta(y_1) = \eta(y_2) = \ldots = \eta(y_i) = \ldots = \eta(y_{r-1})$$

This, however, contradicts the assumption that $\mathcal{H}^{(\mathcal{V})}$ satisfies MCA. □

Analogous to the concepts of credulous and sceptical acceptance, in VAFs the ideas of *subjective* and *objective* acceptance arise.
**Subjective Acceptance** (SBA)
**Instance:** $\mathcal{H}(\mathcal{X}, \mathcal{A}, \mathcal{V}, \eta)$ and $x \in \mathcal{X}$.
**Question:** Is there a specific audience, $\alpha$, for which $x \in P(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle, \alpha)$?
**Objective Acceptance** (OBA)
**Instance:** $\mathcal{H}(\mathcal{X}, \mathcal{A}, \mathcal{V}, \eta)$ and $x \in \mathcal{X}$.
**Question:** Is $x \in P(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle, \alpha)$ for *every* specific audience $\alpha$?

The complexity of SBA and OBA is known to be unchanged under quite extreme restrictions on the form of instances.

**Fact 2** *(Dunne [9])*

1. *Let* SBA$^{(T)}$ *be the decision problem* SBA *with instances restricted to those for which the graph structure* $\langle \mathcal{X}, \mathcal{A} \rangle$ *is a* binary *tree:* SBA$^{(T)}$ *is* NP*–complete.*
2. *Let* SBA$^{(T,\epsilon)}$ *be the decision problem* SBA$^{(T)}$ *in which instances are restricted to those in which* $|\mathcal{V}| \leq |\mathcal{X}|^\epsilon$*:* $\forall\ \epsilon > 0$ SBA$^{(T,\epsilon)}$ *is* NP*–complete.*
3. *Suppose* SBA$^{(\mathcal{V},\leq r)}$ *is the decision problem* SBA *restricted to instances for which* $\forall\ v \in \mathcal{V}\ |\eta^{-1}(v)| \leq r$*, i.e. at most* $r$ *arguments share a common value,* $v \in \mathcal{V}$*. Similarly,* SBA$^{(T),(\mathcal{V},\leq r)}$ *is this problem with instances additionally restricted to trees:* SBA$^{(T),(\mathcal{V},\leq 3)}$ *is* NP*–complete.*

Analogous coNP–completeness results for OBA also hold for the restricted frameworks of Fact 2.

## 2. Algorithms for Subjective and Objective Acceptance

We now describe a general approach by which improvements to the $O(k!|\mathcal{X}|)$ upper bounds on SBA and OBA may be obtained. Underpinning these is the concept of the *value graph* obtained from a VAF. We describe these and their role in algorithms for the decision problems of interest in Section 2.1. In certain cases value graphs allow the set, $\mathcal{U}$, of all specific audiences to be treated in terms of a partition resulting from, what we term, the set of *relevant audiences* with respect to a particular value. Thus, instead of the $k!$ coefficient qualifying algorithmic behaviour, the number of distinct relevant audiences becomes the important factor. In Section 2.2 we establish some upper bounds on this measure for some special classes of value graph.

*2.1. Value graphs*

Given $\mathcal{H}^{(\mathcal{V})}(\mathcal{X}, \mathcal{A}, \mathcal{V}, \eta)$ the *value graph* of $\mathcal{H}^{(\mathcal{V})}$, denoted $\mathcal{G}_{\mathcal{H}}(\mathcal{V}, \mathcal{B})$, is the *directed* graph with vertices $\mathcal{V}$ and edges

$$\mathcal{B} = \{\langle v_i, v_j\rangle \ : \ \exists \ \langle x, y\rangle \in \mathcal{A} \text{ s.t. } \eta(x) = v_i \text{ and } \eta(y) = v_j\} \setminus \{\langle v_i, v_i\rangle \ : \ v_i \in \mathcal{V}\}$$

It should be noted that value graphs exclude so-called self-loops, i.e. directed edges of the form $\langle x, x\rangle$, although in general there will be attacks involving arguments with the same value. To simplify the notation, where no ambiguity arises, we omit the subscript $\mathcal{H}$ and write $\mathcal{G}(\mathcal{V}, \mathcal{B})$.

The idea behind our improved algorithms is to consider structural properties of the *value graph* rather than structural properties of the Dung-style framework described by $\langle \mathcal{X}, \mathcal{A}\rangle$: in view of Fact 2 the latter approach appears unpromising as a source of efficient methods.

Before introducing these forms, it will be helpful to introduce the following notation which aids in relating properties of value graphs to properties of subsets of arguments in $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta\rangle$.

**Definition 4** *Given $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta\rangle$ with $\mathcal{V} = \{v_1, v_2, \ldots, v_k\}$, the* AF *induced by $v_j$, denoted $\mathcal{H}_j(\mathcal{X}_j, \mathcal{A}_j)$ has $\mathcal{X}_j = \{ x \in \mathcal{X} \ : \ \eta(x) = v_j\}$ and $\mathcal{A}_j = \{\langle x, y\rangle \ : \ \eta(x) = \eta(y) = v_j\} \cap \mathcal{A}$. More generally, the* AF *induced by a subset, $W$ of $\mathcal{V}$ – $\mathcal{H}_W(\mathcal{X}_W, \mathcal{A}_W)$ has*

$$\begin{aligned} \mathcal{X}_W &= \{x \in \mathcal{X} \ : \ \eta(x) \in W\} \\ \mathcal{A}_W &= \{\langle x, y\rangle \ : \ \{\eta(x), \eta(y)\} \subseteq W\} \cap \mathcal{A} \end{aligned}$$

The two principal forms of value graph we consider are *strict trees* and *chains* the latter being a subclass of the former.

**Definition 5** *A value graph, $\mathcal{G}(\mathcal{V}, \mathcal{B})$ is a* strict tree *if the* undirected *graph formed by replacing each directed edge $\langle v_i, v_j\rangle$ with the undirected edge $\{v_i, v_j\}$ is acyclic, i.e. defines a* tree*.*

An important subclass of strict trees is the class of *chain graphs*

**Definition 6** *A value graph, $\mathcal{G}(\mathcal{V}, \mathcal{B})$ is a* chain *if it satisfies:*

a. *$\mathcal{G}(\mathcal{V}, \mathcal{B})$ is a strict tree.*
b. *The undirected graph resulting from $\mathcal{G}(\mathcal{V}, \mathcal{B})$ (as described in Defn. 5) forms a simple path joining all vertices in $\mathcal{V}$.*

Figure 1 gives examples of strict trees and chains defined over the value set $\mathcal{V} = \{A, B, C, D, E, F\}$. For a vertex $v$ in a strict tree the set of vertices $\{ w_i \ : \ \langle w_i, v\rangle \in \mathcal{B}\}$ are referred as the *children* of $v$ (similarly $v$ is called the *parent* of $w_i$). The notation $ch(v)$ will be used for the set of children of $v$ and $par(w)$ for the parent of $w$.

The *sub-tree* of $\mathcal{G}$ *rooted at* $v$, denoted $\langle W_v, F_v\rangle$, is recursively defined as follows:

a. If $v$ has no children, i.e. $ch(v) = \emptyset$, then $W_v = \{v\}$ and $F_v = \emptyset$.

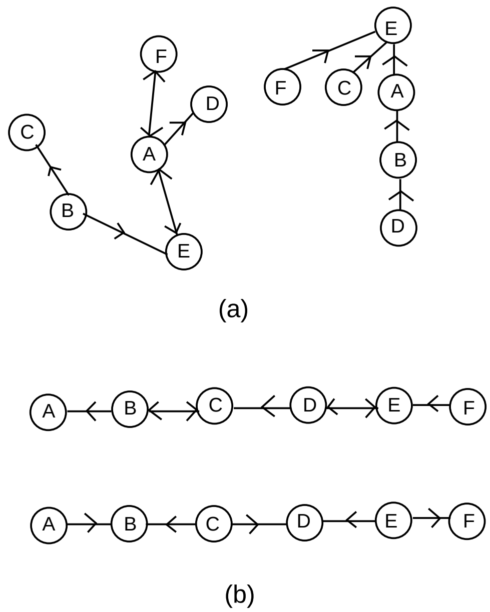

**Figure 1.** (a) Strict tree examples. (b) Two Chain graphs.

b. Otherwise, let $ch(v) = \{w_1, \ldots, w_r\}$. In this case,

$$W_v = \{v\} \cup \bigcup_{u \in ch(v)} W_u \quad ; \quad F_v = \bigcup_{u \in ch(v)} \{\langle u, v\rangle\} \cup F_u$$

The *height* of $w \in \langle W_v, F_v\rangle$ is denoted $ht(v)$. This is defined to be 0 when $ch(v) = \emptyset$ and $1 + \max_{w \in ch(v)} ht(w)$ otherwise.

We note the following, easily proven, property of the sub-tree $\langle W_v, F_v\rangle$ arising from the strict tree $\mathcal{G}(\mathcal{V}, \mathcal{B})$:

Given a value graph, $\mathcal{G}(\mathcal{V}, \mathcal{B})$ defining a strict tree, the sub-tree rooted at $v$ induces a partial order, $\sqsubseteq_F$ over $W_v$ defined via $v_i \sqsubseteq_F v_j$ if $i = j$ or there is a sequence $v_i = v_0 v_1 \cdots v_r = v_j$ of *distinct* values such that $\langle v_t, v_{t+1}\rangle \in F_v$ for each $0 \leq t < r$.

Let $\mu_F$ denote the set of *minimal* elements in this partial order. The second example from Fig 1(a) gives rise to the partial order

$$C \sqsubseteq E \;\; ; \;\; D \sqsubseteq B \sqsubseteq A \sqsubseteq E \;\; ; \;\; F \sqsubseteq E$$

With the sets of minimal elements being $\{C, D, F\}$

It is not hard to show that,

a. $w \in W_v$ if and only if $w \sqsubseteq_F v$.
b. The root vertex $v$ is the unique maximal element w.r.t. $\sqsubseteq_F$ among the vertices $W_v$.
c. If $w$ is not the root vertex than $par(w)$ contains exactly one vertex (in $\langle W_v, F_v\rangle$), otherwise $par(w) = \emptyset$.

Finally we note the following important property of (strict) trees: given any $u \in W_v$ there is a unique sequence, $\delta(u,v)$, of *distinct* directed edges in $F_v$ with

$$\delta(u,v) \quad = \quad \langle u, u_1\rangle \ \cdot \ \langle u_1, u_2\rangle \ \cdot \ \langle u_2, u_3\rangle \ \cdots \ \langle u_{r-1}, u_r\rangle \ \cdot \ \langle u_r, v\rangle$$

That is, there is a unique path from $u$ to $v$ within $F_v$. When $u = v$ this is the empty sequence.

A key idea underpinning our improved algorithms is the concept of the set of *relevant audiences* for a sub-tree rooted at $v$. We give the definition below and discuss its application subsequently.

**Definition 7** *Let $\langle W_v, F_v\rangle$ be the sub-tree rooted at $v$ of a strict tree $\mathcal{G}(\mathcal{V},\mathcal{B})$. Let $R \subseteq \mathcal{V}\times\mathcal{V}$ be an audience. We say that $R$ is* relevant with respect to $v$ *if*

$$p \succ_R q \quad \Rightarrow \quad \langle q,p\rangle \in F_v \quad \textit{and} \quad \forall\ \langle x,y\rangle \in W_q \times W_q\ \neg(x \succ_{R^*} y)$$

*The* strict tree induced in $\langle W_v, F_v\rangle$ *by $R$, denoted $\langle \mathcal{V}_R, \mathcal{E}_R\rangle$, has*

$$\mathcal{V}_R \quad = \quad \{\ u \in W_v\ :\ \forall\ \langle p,q\rangle \in \delta(u,v)\ \neg(q \succ_R p)\ \}$$

$$\mathcal{E}_R \quad = \quad F_v\ \cap\ \{\langle p,q\rangle\ :\ p \in \mathcal{V}_R,\ q \in \mathcal{V}_R\}$$

*For $w \in \mathcal{V}_R$ of the strict tree induced by $R$, we use $ch(w,R)$ to denote the set of children of $w$ in the strict tree $\langle \mathcal{V}_R, \mathcal{E}_R\rangle$, i.e.*

$$ch(w,R) \quad = \quad \{\ u\ :\ \langle u,w\rangle \in \mathcal{E}_R\}$$

*We note that $ch(w,R)$ may be a strict subset of $ch(w)$ the children of $w$ in the strict tree $\langle W_v, F_v\rangle$. The* height of $w$ *in $\langle \mathcal{V}_R, \mathcal{E}_R\rangle$, $ht(w,R)$, is defined as*

$$ht(w,R) \quad = \quad \begin{cases} 0 & \textit{if } ch(w,R) = \emptyset \\ 1 + \max_{u \in ch(w,R)}\ ht(u,R) & \textit{otherwise} \end{cases}$$

For example, in the case of the second example from Fig. 1(a) there are exactly 16 relevant audiences with respect to $E$. These are

$\emptyset$
$\{E \succ A\},\ \{A \succ B\},\ \{B \succ D\}$
$\{E \succ F\},\ \{E \succ F, E \succ A\},\ \{E \succ F, A \succ B\},\ \{E \succ F, B \succ D\}$
$\{E \succ C\},\ \{E \succ C, E \succ A\},\ \{E \succ C, A \succ B\},\ \{E \succ C, B \succ D\}$
$\{E \succ F, E \succ C\},\ \{E \succ F, E \succ C, E \succ A\}$
$\{E \succ F, E \succ C, A \succ B\},\ \{E \succ F, E \succ C, B \succ D\}$

Some of the resulting strict trees induced are shown in Fig. 2

For a value $v \in \mathcal{V}$ we denote by $\mathcal{R}_v$ the set of relevant audiences with respect to $v$, i.e.

$$\mathcal{R}_v \quad = \quad \{\mathcal{R} \subset W_v \times W_v\ :\ \mathcal{R} \text{ is relevant w.r.t. } v\}$$

Given $R \in \mathcal{R}_v$ with $\langle \mathcal{V}_R, \mathcal{E}_R\rangle$ the strict tree induced in $\langle W_v, F_v\rangle$ by $R$ we define the *framework associated with* $R$, denoted $\langle \mathcal{Z}_R, \mathcal{D}_R\rangle$, as the AF with

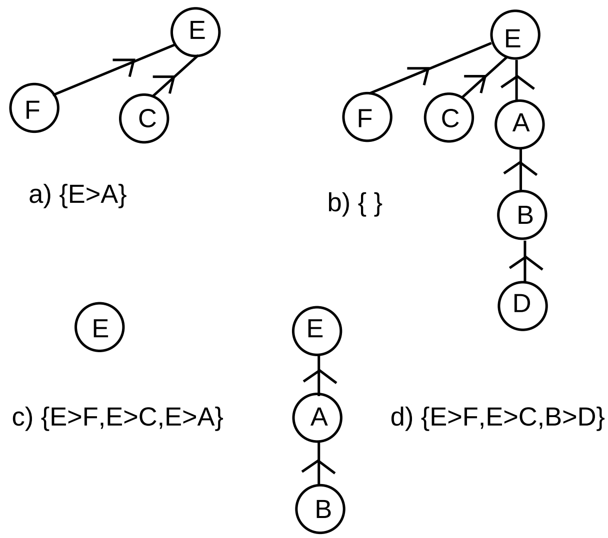


**Figure 2.** Strict trees induced by relevant audiences

$$\mathcal{Z}_R \;=\; \bigcup_{w\in\mathcal{V}_R} \{\; x \;:\; \eta(x) = w\}$$

$$\mathcal{D}_R \;=\; (\mathcal{A} \cap \{\langle x, y\rangle \;:\; x \in \mathcal{Z}_R,\; y \in \mathcal{Z}_R\}) \setminus \{\langle x, y\rangle \;:\; \eta(y) \succ_R \eta(x)\}$$

It may be noted that $\langle \mathcal{Z}_R, \mathcal{D}_R\rangle$ is a *sub-graph* of AF induced by $\mathcal{V}_R$.

**Lemma 1** *Let $\mathcal{H}^{(\mathcal{V})}(\mathcal{X}, \mathcal{A}, \mathcal{V}, \eta)$ be a* VAF *whose value graph, $\mathcal{G}(\mathcal{V}, \mathcal{B})$ defines a strict tree. For every $v \in \mathcal{V}$ and every $R \in \mathcal{R}_v$ the framework, $\langle \mathcal{Z}_R, \mathcal{D}_R\rangle$, associated with $R$ is acyclic and thus has a unique (non-empty) preferred extension, $P(\mathcal{Z}_R, \mathcal{D}_R)$.*

**Proof:** Let $\mathcal{H}^{(\mathcal{V})}(\mathcal{X}, \mathcal{A}, \mathcal{V}, \eta)$ satisfy the premises of the Lemma statement. Consider any $v \in \mathcal{V}$ and $R \in \mathcal{R}_v$ letting $\langle \mathcal{Z}_R, \mathcal{D}_R\rangle$ be the framework associated with $R \in \mathcal{R}_v$. Suppose, to the contrary, that there is a sequence of $r$ (distinct) arguments $z_0 z_1 z_2 \cdots z_r$ in $\mathcal{Z}$ for which $z_r = z_0$ and $\langle z_i, z_{i+1}\rangle \in \mathcal{D}_R$ for each $0 \leq i < r$. Consider the sequence of *values* defined by this cycle, i.e. $\eta(z_0)\eta(z_1) \cdots \eta(z_{r-1})\eta(z_r)$. By definition $\eta(z_r) = \eta(z_0)$, and there is at least one argument, $z_i$, on this cycle for which $\eta(z_i) \neq \eta(z_0)$. Furthermore, since all of the attacks $\langle z_i, z_{i+1}\rangle$ are present in $\mathcal{D}_R$, we must have $\{z_0, z_1, \ldots, z_r\} \subseteq \mathcal{Z}_R$ and $\neg(\eta(z_{i+1}) \succ_R \eta(z_i))$ for each $0 \leq i < r$. From the definition of value graph, we deduce that $\mathcal{G}(\mathcal{V}, \mathcal{B})$ contains *both* a directed path from $v_{\eta(z_0)}$ to $v_{\eta(z_i)}$ *and* a directed path from $v_{\eta(z_i)}$ to $v_{\eta(z_0)}$. This, however, contradicts the assumption that $\mathcal{G}(\mathcal{V}, \mathcal{B})$ is a strict tree.

Having established that the framework associated with $R$ is acyclic, it follows that this has a unique, non-empty preferred extension. □

**Definition 8** *Let $R$ be a relevant audience w.r.t. $v$ in $\langle W_v, F_v\rangle$, and $\langle \mathcal{V}_R, \mathcal{E}_R\rangle$ the strict tree induced by $R$. A specific audience, $\alpha$, is said to be $R$-*compatible *if*

S1. $\forall\ \langle p,q\rangle \in \mathcal{E}_R \quad p \succ_\alpha q.$
S2. $\forall\ \langle p,q\rangle \in F_v \quad (q \succ_R p) \quad \Rightarrow \quad (q \succ_\alpha p)$

*We denote by $\chi(R,v)$ the set of specific audiences that are $R$-compatible.*

*For $\alpha$ a specific audience, we say that a relevant audience w.r.t. $v$ is $\alpha$-*compatible *if*

R1. $\forall\ \langle p,q\rangle \in F_v \quad (q \succ_\alpha p) \ \Rightarrow\ q \succ_R p \quad \textit{and}\ \forall\ \langle x,y\rangle \in W_p \times W_p\ \neg(x \succ_{R^*} y)$
R2. $\forall\ \langle p,q\rangle \in \mathcal{E}_R\ p \succ_\alpha q.$

*We use $\rho(\alpha,v)$ to denote the set of relevant audiences w.r.t. $v$ that are $\alpha$-compatible.*

It is straightforward, using these definitions, to show that,

$$\forall\ \alpha \in \mathcal{U},\ \forall\ R \in \mathcal{R}_v \quad \alpha \in \chi(R,v) \quad \Leftrightarrow \quad R \in \rho(\alpha,v)$$

We need two key properties of the set of relevant audiences w.r.t. $v$ in obtaining improved algorithms for SBA and OBA. The first – Thm. 1 – shows that $\mathcal{R}_v$ induces a partition of $\mathcal{U}$; the second, presented in Thm. 2, establishes that it suffices to consider only the *partial* (i.e. not specific) audiences represented in $\mathcal{R}_v$ when considering acceptance properties of arguments with value $v$.

**Theorem 1** *For $\langle W_v, F_v\rangle$ the sub-tree of $\mathcal{G}(\mathcal{V},\mathcal{B})$ with root $v \in \mathcal{V}$, the set of relevant audience audiences w.r.t. $v$, $\mathcal{R}_v$, satisfies all of the following properties:*

a. $\forall\ R \in \mathcal{R}_v \quad \chi(R,v) \neq \emptyset$, *i.e. there is* at least *one $R$-compatible specific audience for each $R \in \mathcal{R}_v$.*
b. *For every $\alpha \in \mathcal{U}$ there is some $R \in \mathcal{R}_v$ for which $\alpha$ is $R$-compatible. Formally*

$$\bigcup_{R \in \mathcal{R}_v} \chi(R,v) \quad = \quad \mathcal{U}$$

c. *Given $R$ and $S$ in $\mathcal{R}_v$, the sets of $R$-compatible specific audiences are disjoint from the set of $S$-compatible specific audiences, i.e.*

$$\forall\ R,\ S \in \mathcal{R}_v \quad \chi(R,v) \cap \chi(S,v) \neq \emptyset \quad \Leftrightarrow \quad R = S$$

**Proof:** For part (a), given $R \in \mathcal{R}_V$ with induced sub-tree $\langle \mathcal{V}_R, \mathcal{E}_R\rangle$ any specific audience, $\alpha$, for which $p \succ_\alpha q$ whenever $p \succ_R q$ or $\langle p,q\rangle \in \mathcal{E}_R$ is in $\chi(R,v)$. Noting that $p \succ_R q$ implies $\langle q,p\rangle \in F_v$ leading to $q \notin \mathcal{V}_R$, this construction means that for every $\langle p,q\rangle \in F_v$ exactly one of $p \succ_\alpha q$ or $q \succ_\alpha p$ will hold.

For part (b), it suffices to show that for every $\alpha \in \mathcal{U}$, $\rho(\alpha,v) \neq \emptyset$. This is easily seen using $R \subseteq W_v \times W_v$ defined as follows. First form the set

$$S \quad = \quad \{\langle p,q\rangle \in W_v \times W_v \ :\ \langle q,p\rangle \in F_v \text{ and } p \succ_\alpha q\}$$

We now obtain $R \in \rho(\alpha,v)$ via

$$p \succ_R q \quad \Leftrightarrow \quad \langle p,q\rangle \in S \setminus \{\langle u,w\rangle \ :\ \{u,w\} \subseteq W_t \text{ and } par(t) \succ_\alpha t\}$$

It is straightforward to check that $R$ defined in this way is a relevant audience w.r.t. $v$ and satisfies the conditions R1 and R2 for $\alpha$-compatability.

Finally for part (c), that $\chi(R,v)\cap\chi(S,v)\neq\emptyset$ whenever $R=S$ is self-evident. So assume, to the contrary, that $\alpha\in\chi(R,v)\cap\chi(S,v)$ for distinct $R\in\mathcal{R}_v$ and $S\in\mathcal{R}_v$. Since $R\neq S$, without loss of generality, there is some $\langle x,y\rangle\in F_v$ for which $y\succ_R x$ but $\neg(y\succ_S x)$. Consider the strict trees $\langle\mathcal{V}_R,\mathcal{E}_R\rangle$ and $\langle\mathcal{V}_S,\mathcal{E}_S\rangle$ induced by $R$ and $S$. From $y\succ_R x$ and $\langle x,y\rangle\in F_v$ we must have $y\succ_\alpha x$ via condition S2 for $\alpha\in\chi(R,v)$. On the other hand, from $\neg(y\succ_S x)$ we must have $x\in\mathcal{V}_S$ since the path $\langle x,y\rangle\cdot\delta(y,v)$ is preserved in $\mathcal{E}_S$. We now obtain a contradiction: from condition S1, since $\alpha\in\chi(S,v)$ and $\langle x,y\rangle\in\mathcal{E}_S$ therefore $y\succ_\alpha x$ and $\alpha$ cannot satisfy both $x\succ_\alpha y$ as results from $\alpha\in\chi(R,v)$ and $y\succ_\alpha x$ which results from $\alpha\in\chi(S,v)$. □

**Corollary 1** *Given $\mathcal{H}^{(\mathcal{V})}(\mathcal{X},\mathcal{A},\mathcal{V},\eta)$ whose value graph is a strict tree, let $\langle W_v,F_v\rangle$ be the sub-tree with root $v$. For all specific audiences, $\alpha$, $\rho(\alpha,v)$ contains* exactly one *relevant audience w.r.t. $v$.*

**Proof:** Immediate from Thm. 1: for $R\in\rho(\alpha,v)$ we have $\alpha\in\chi(R,v)$ but from Thm. 1(c) there is exactly one $R\in\mathcal{R}_v$ for which $\alpha\in\chi(R,v)$. □

Recalling that $\langle\mathcal{X}_v,\mathcal{A}_v\rangle$ is the AF induced by arguments with value $v$, we now establish the second key property of relevant audiences w.r.t. $v$.

**Theorem 2** *For all $v\in\mathcal{V}$, $R\in\mathcal{R}_v$ and $\alpha\in\chi(R,v)$*

$$\mathcal{X}_v\cap P(\mathcal{Z}_R,\mathcal{D}_R)\ =\ \mathcal{X}_v\cap P(\mathcal{H}^{(\mathcal{V})},\alpha)$$

**Proof:** We proceed by induction on $ht(v)$, to prove $Q(t)$, where $Q(t)$ is the statement

"If $ht(v)=t$ then for all $R\in\mathcal{R}_v$ and $\alpha\in\chi(R,v)$, $\mathcal{X}_v\cap P(\mathcal{Z}_R,\mathcal{D}_R)=\mathcal{X}_v\cap P(\mathcal{H}^{(\mathcal{V})},\alpha)$"

**Base:** $t=0$, so that $ht(v)=0$
When $ht(v)=0$ we have $ch(v)=\emptyset$ so that $\mathcal{R}_v=\{\emptyset\}$ and $\chi(\emptyset,v)=\mathcal{U}$. In consequence, $\mathcal{Z}_R=\mathcal{X}_v$ and $\mathcal{D}_R=\mathcal{A}_v$. Again from $ch(v)=\emptyset$ we deduce that if $y\in\mathcal{X}_v^-$ then $\eta(y)=v$. It now follows that every argument in $P(\mathcal{X}_v,\mathcal{A}_v)$ is objectively accepted: regardless of the value ordering of $\alpha\in\mathcal{U}$ no attacks on $\mathcal{X}_v$ will be altered, thus $P(\mathcal{X}_v,\mathcal{A}_v)$ will be a subset of $P(\mathcal{H}^{(\mathcal{V})},\alpha)$ as required.
**Inductive Step:** Assuming $Q(s)$ for all $s\leq t-1$, we show that $Q(t)$ holds.
Suppose $ht(v)=t\geq 1$ in $\langle W_v,F_v\rangle$. Let $ch(v)=\{w_1,w_2,\ldots,w_r\}$. From the inductive hypothesis, since $ht(w)\leq t-1$ for each $w\in ch(v)$, we know that

$$\forall\ R\in\mathcal{R}_w\ \forall\alpha\in\chi(R,w)\ \ \mathcal{X}_w\cap P(\mathcal{Z}_R,\mathcal{D}_R)\ =\ \mathcal{X}_w\cap P(\mathcal{H}^{(\mathcal{V})},\alpha)$$

Consider any $R\in\mathcal{R}_v$. From the definition of relevant audience, it follows that this is characterised by some subset $S=\{s_1,s_2,\ldots,s_q\}$ of $ch(v)$ and, for each member $s_i$ of $S$, a relevant audience, $S_i$ w.r.t. to $s_i$ in $\langle W_{s_i},F_{s_i}\rangle$ so that

$$R \;=\; \bigcup_{i=1}^{q} S_i \;\cup\; \bigcup_{t \in ch(v)\setminus S} \{v \succ_R t\}$$

Furthermore, for $R \in \mathcal{R}_v$ defined from $\langle S_1, S_2, \ldots, S_q\rangle$ as described, $\chi(R, v)$ is

$$\bigcap_{i=1}^{q} \chi(S_i, s_i) \;\cap\; \bigcap_{s \in S} \{\, \alpha \;:\; s \succ_\alpha v\} \;\cap\; \bigcap_{t \in ch(v)\setminus S} \{\, \alpha \;:\; v \succ_\alpha t\}$$

so that $\chi(R, v) \subseteq \chi(S_i, s_i)$ for all $1 \leq i \leq q$.

Now let $\langle S_1, \ldots, S_q\rangle$ be the relevant audiences from the subset $S$ of $ch(v)$ used in forming $R$. From the inductive hypothesis we know that for each $j$ ($1 \leq j \leq q$) and each $\alpha \in \chi(S_j, s_j)$,

$$\mathcal{X}_{s_j} \cap P(\mathcal{Z}_{S_j}, \mathcal{D}_{S_j}) = \mathcal{X}_{s_j} \cap P(\mathcal{H}^{(\mathcal{V})}, \alpha)$$

Noting that

$$\mathcal{Z}_R \;=\; \mathcal{X}_v \;\cup\; \bigcup_{s_i \in S} \mathcal{Z}_{S_i}$$

let $P_j \;=\; P(\mathcal{Z}_{S_j}, \mathcal{D}_{S_j}) \cap \mathcal{X}_{s_j}$. Then

$$P_j \;=\; \mathcal{X}_{s_j} \cap P(\mathcal{Z}_{S_j},\, \mathcal{D}_{S_j}) \;=\; \mathcal{X}_{s_j} \cap P(\mathcal{H}^{(\mathcal{V})}, \alpha) \;\; \forall\, \alpha \in \chi(S_j, s_j)$$

via the inductive hypothesis, and hence $P_j \;=\; \mathcal{X}_{s_j} \cap P(\mathcal{H}^{(\mathcal{V})}, \alpha)$ for all $\alpha \in \chi(R, v)$. It follows that, without loss of generality, in determining $P(\mathcal{Z}_R, \mathcal{D}_R) \cap \mathcal{X}_v$ we need only consider the AF $\langle \mathcal{Y}_R, \mathcal{C}_R\rangle$ in which

$$\mathcal{Y}_R \;=\; \mathcal{X}_v \;\cup\; \bigcup_{s_j \in S} P_j$$

$$\begin{aligned}\mathcal{C}_R \;=\; &\{\langle x, y\rangle \;:\; x \in \mathcal{X}_v,\; y \in \mathcal{X}_v\} \cap \mathcal{A} \;\cup \\ &\{\langle x, y\rangle \;:\; \eta(y) = v,\; \eta(x) = s_j,\; x \in P_j\} \cap \mathcal{A}\end{aligned}$$

and in this AF we have

$$\mathcal{X}_v \cap P(\mathcal{Y}_R, \mathcal{C}_R) \;=\; \mathcal{X}_v \cap P(\mathcal{H}^{(\mathcal{V})}, \alpha) \;\; \forall\, \alpha \in \chi(R, v)$$

so that

$$\mathcal{X}_v \cap P(\mathcal{Z}_R, \mathcal{D}_R) \;=\; \mathcal{X}_v \cap P(\mathcal{H}^{(\mathcal{V})}, \alpha) \;\; \forall\, \alpha \in \chi(R, v)$$

competing the inductive proof. □

**Corollary 2** *Let $\mathcal{H}^{(\mathcal{V})}(\mathcal{X}, \mathcal{A}, \mathcal{V}, \eta)$ be a* VAF *whose value graph defines a strict tree. Let $x \in \mathcal{X}$ be any argument and $\eta(x) = v \in \mathcal{V}$.*

a. SBA$(\mathcal{H}^{(\mathcal{V})}, x)$ *if and only if there exists some $R \in \mathcal{R}_v$, i.e. a relevant audience with respect to $\eta(x)$, for which $x \in P(\mathcal{Z}_R, \mathcal{D}_R)$*
b. OBA$(\mathcal{H}^{(\mathcal{V})}, x)$ *if and only if for* every, *$R \in \mathcal{R}_v$, $x \in P(\mathcal{Z}_R, \mathcal{D}_R)$.*

**Proof:** For part(a), if $x$ with $\eta(x) = v$ is subjectively accepted in $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$, consider the specific audience, $\alpha$, witnessing this property. From Corollary 1 and Thm. 1 we identify $R \in \mathcal{R}_v$ for which $\alpha \in \chi(R, v)$. Thus, from Thm. 2, $x \in P(\mathcal{Z}_R, \mathcal{D}_R)$ as required. Conversely, if $x \in P(\mathcal{Z}_R, \mathcal{D}_R)$ for some $R \in \mathcal{R}_v$ then we for any any $\alpha \in \chi(R, v)$ we have $x \in P(\mathcal{H}^{(\mathcal{V})}, \alpha)$ so that $x$ is subjectively accepted.

For part (b), should $x$ be objectively accepted then $x \in P(\mathcal{H}^{(\mathcal{V})}, \alpha)$ for every specific audience and hence, via Thms. 1 and 2, $x \in P(\mathcal{Z}_R, \mathcal{D}_R)$ for every $R \in \mathcal{R}_v$. Similarly, if $x \in P(\mathcal{Z}_R, \mathcal{D}_R)$ for every $R \in \mathcal{R}_v$ then, from Thm. 2, $x \in P(\mathcal{H}^{(\mathcal{V})}, \alpha)$ for every $\alpha \in \chi(R, v)$, and from Thm. 1 every specific audience is $R$-compatible for some $R \in \mathcal{R}_v$ so that $x \in P(\mathcal{H}^{(\mathcal{V})}, \alpha)$ for every specific auidence, i.e. $x$ is objectively accepted. □

In total, Thms. 1, 2 and Corollary 2 suggest the approach of Algorithm 2 for deciding SBA and OBA.

The *orientation* of $\mathcal{G}(\mathcal{V}, \mathcal{B})$ with respect to $\sqsubseteq_F$ is recursively defined via the process of Algorithm 1 so that each $v \in \mathcal{V}$ is assigned a label (called its *orientation level*) denoted $\omega(v)$.

**Algorithm 1** Assigning Orientation Levels to Strict Trees

**for** $v \in \mu_F$ **do**
  $\omega(v) := 0$
**end for**
**for** $v \in \mathcal{V} \setminus \mu_F$ **do**
  $\omega(v) := \perp$
**end for**
**while** $\exists\, w \in \mathcal{V} \,:\, \omega(w) := \perp$ **do**
  Choose any such $w$ having $\omega(v) \neq \perp$ for all $v \in ch(w)$.
  $\omega(w) := 1 + \max_{v \in ch(w)} \omega(v)$
**end while**

Notice that each of the sets $\langle R, \langle \mathcal{Z}_R, \mathcal{D}_R \rangle \rangle$ need be computed only once (and subsequently stored). Via Thm. 2, any $R \in \mathcal{R}_v$ is uniquely described from $S = \{s_1, \ldots, s_q\} \subseteq ch(v)$, by

$$\bigcup_{i=1}^{q} S_i \;\cup \bigcup_{t \in ch(v) \setminus S} \{v \succ t\}$$

where $S_i \in \mathcal{R}_{s_i}$.

Noting that $P(\mathcal{Z}_R, \mathcal{D}_R)$ can be obtained in $O(|\mathcal{Z}_R|)$ steps, the run-time of Alg. 2 is bounded above by $O(|\mathcal{R}_v| \times |\mathcal{X}|)$. Hence in structures for which $|\mathcal{R}_v|$ can be guaranteed to be polynomial in $|\mathcal{V}|$, Alg. 2 provides a polynomial time decision process for subjective and objective acceptance in VAFs.

### *2.2. Bounding the number of relevant audiences*

In this section we consider the behaviour of the function $r : \langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle \rightarrow \mathbb{N}$ with $|\mathcal{V}| = k$, $\mathcal{G}(\mathcal{V}, \mathcal{B})$ defining a strict tree, and $r(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle)$ defined as

**Algorithm 2** Deciding argument status in strict tree VAFs

**function** STATUS($\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle,\ x \in \mathcal{X}$)
$\langle W_{\eta(x)},\ F_{\eta(x)} \rangle$ := strict tree with root $\eta(x)$
Orientate $\langle W_{\eta(x)},\ F_{\eta(x)} \rangle$ using Alg. 1.
Mark each $u \in W_{\eta(x)}$ as **unprocessed**.
**while** $\exists\ u \in W_{\eta(x)}\ :\ u$ *is* **unprocessed do**
  **while** $\exists\ w \in W_{\eta(x)}\ :\ \omega(w) = 0$ *and* $w$ *is* **unprocessed do**
    Compute $P(\mathcal{X}_w, \mathcal{A}_w)$
    Mark $w$ as **processed**.
  **end while**
  Choose any $w$ which is **unprocessed** and $\forall\ u \in ch(w)\ u$ is **processed**.
  **for** $R \in \mathcal{R}_w$ **do**
    Compute $P(\mathcal{Z}_R, \mathcal{D}_R)$
  **end for**
  Mark $w$ as **processed**
**end while**
if $x \in P(\mathcal{Z}_R, \mathcal{D}_R)$ for every $R \in \mathcal{R}_{\eta(x)}$ report OBA($x$)
if $x \in P(\mathcal{Z}_R, \mathcal{D}_R)$ for some $R \in \mathcal{R}_{\eta(x)}$ report SBA($x$)
else report $x$ is indefensible.

$$r(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle) \quad = \quad \max_{v \in \mathcal{V}}\ |\mathcal{R}_v|$$

In terms of Alg. 2, we obtain algorithms with run-time $O(r(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle)|\mathcal{X}|^2)$.

We first show that when the value graph is a strict tree we always obtain an improvement on $k!$

**Lemma 2** *For any* VAF, $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$ *whose value graph is a strict tree,*

$$r(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle) \quad \leq \quad 2^{k-1}$$

**Proof:** Let $\langle W_v, F_v \rangle$ be the strict tree defined by the value graph $\mathcal{G}(\mathcal{V}, \mathcal{B})$. Then $|\mathcal{R}_w|$ is maximised when $w = v$ the root of this tree. Any $R \in \mathcal{R}_v$ maps to a subset of $F_v$ so that $|\mathcal{R}_v|$ cannot exceed the total number of subsets of $F_v$.[3] Since $\langle W_v, F_v \rangle$ is a strict tree and $|W_v| \leq |\mathcal{V}| = k$, it follows that $|F_v| = k - 1$ giving the upper bound claimed. □

Although, as we shall see, the upper bound of Lemma 2 overestimates $|\mathcal{R}_v|$ there are cases where this bound cannot be improved.

**Lemma 3** *Let* $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$ *be such that* $\langle W_v, F_v \rangle$ *consists of a root vertex* $v$ *with* $ch(v) = \mathcal{V} \setminus \{v\}$. *Then* $r(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle) = 2^{k-1}$.

**Proof:**

$$r(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle) = \sum_{S \subseteq ch(v)} \prod_{t \in S} |\mathcal{R}_t| = \sum_{S \subseteq ch(v)} 1 \ = \ 2^{|ch(v)|} \ = \ 2^{k-1}$$

[3]Recall that $F_v$ will only contain directed edges in $\delta(u, v)$ for some $u \in W_v$ thus $|F_v| = |W_v| - 1 \leq k - 1$.

□

When $\mathcal{G}(\mathcal{V},\mathcal{B})$ is a chain graph we obtain a further significant improvement

**Theorem 3** *If the value graph $\mathcal{G}(\mathcal{V},\mathcal{B})$ of $\langle\mathcal{X},\mathcal{A},\mathcal{V},\eta\rangle$ is a chain then*

$$r(\langle\mathcal{X},\mathcal{A},\mathcal{V},\eta\rangle) \quad \leq \quad \begin{cases} \frac{(k-1)^2}{4} & \textit{if } k \textit{ is odd} \\ \\ \frac{k(k-2)}{4} & \textit{if } k \textit{ is even} \end{cases}$$

**Proof:** Suppose $v \in \mathcal{V}$ maximises $r(\langle\mathcal{X},\mathcal{A},\mathcal{V},\eta\rangle)$, i.e. is the root of the strict tree arising from $\mathcal{G}(\mathcal{V},\mathcal{B})$. If $ch(v) = \{w\}$ then $|\mathcal{R}_v| = 1 + |\mathcal{R}_w|$, thus we can assume that $ch(v) = \{u, w\}$ so that

$$|\mathcal{R}_v| \quad = \quad 1 \ + \ |\mathcal{R}_u| \ + \ |\mathcal{R}_w| \ + \ |\mathcal{R}_u| \times |\mathcal{R}_w| \quad = \quad (1 + |\mathcal{R}_u|) \times (1 + |\mathcal{R}_w|)$$

The sub-trees with roots $u$ and $w$ must consist of single directed paths from (distinct) vertices with no children and exactly one parent (in order to maximise $|\mathcal{R}_v|$). Letting $k_u$ denote $|W_u|$ and $k_w = |W_w|$ so that $k_u + k_w = k - 1$, it follows that

$$|\mathcal{R}_v| \quad \leq \quad \max_{k_u,\ k_w\ :\ k_u+k_w=k-1} \quad (1 + k_u)(1 + k_w)$$

When $k$ is the odd these are achieved by $k_u = k_w = (k-1)/2$; when $k$ is even by $k/2$ and $(k-2)/2$. □

In fact, Thm. 3, is alternatively derived as the special case $t = 1$, $d = 2$ of a more general upper estimate of the maximum size of $|\mathcal{R}_v|$ that we now consider. Let $e(k,t,d)$ be defined as

$$\max_{\langle W_v,F_v\rangle} \ \{|\mathcal{R}_v| \ : \ |W_v| = k, \ \max_{w\in W_v} \ |ch(w)| = d, \ |\{w \in W_v \ : \ |ch(w)| > 1\}| = t\}$$

In the following theorem our aim is not to prove exact bounds on $e(k,t,d)$ but rather to give an indication of the circumstances where polynomial time algorithms for SBA and OBA would result from Alg. 2.

**Theorem 4**

$$\text{a.} \qquad e(k,0,1) \quad = \quad e(k,0,d) \quad = \quad k$$

$$\text{b.} \qquad e(k,t,d) \ \leq \ \frac{(t(d-2)+k+1)^{td}}{(k-t)^{t-1} \times (t(d-1)+1)^{t(d-1)+1}}$$

**Proof:** (outline) Part (a) is an easy induction on $k \geq 1$. For part (b) we make use of the following: in bounding $e(k,t,d)$ it suffices to consider strict trees $\langle W_v, F_v\rangle$ for which $|ch(v)| = d$ and in which the sub trees whose roots are the children of $v$ maximise, $e((k-1)/d, (t-1)/d, d)$ (we omit the rather tedious proof of this fact). In other words, if there are $t$ values in $W_v$ having $d$ children, it may be assumed that these values form a $d$-ary tree, i.e. with $d^r$ leaf nodes, each such leaf being a chain of $(k-t)/d^r$ values as illustrated in Fig. 3.

Thus, $e(k,t,d)$ is maximised when $\langle W_v, F_v\rangle$ is formed as a $t$ vertex $d$-ary tree in which each leaf is replaced by a chain of $(k-t)/(t(d-1)+1)$ values. This leads

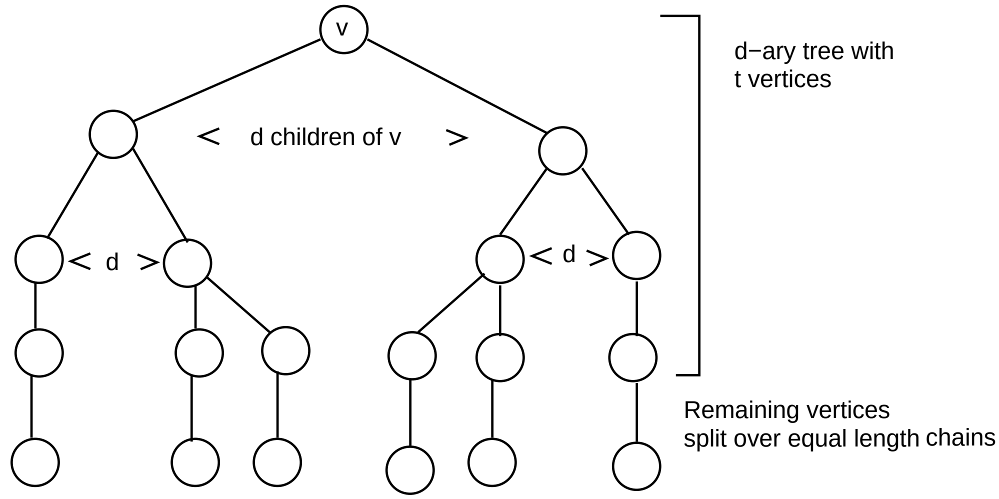


**Figure 3.** Strict tree maximising $e(k,t,d)$

to the recurrence relation, $e(k,t,d) \leq \alpha(r)$ where $r = \log_d(t(d-1)+1)$ and $\alpha(r)$ satisfies

$$\alpha(0) = \tfrac{k-t}{t(d-1)+1}$$

$$\alpha(r) = (1 + \alpha(r-1))^d$$

The general case, $\alpha(r)$ when $r > 0$, is derived from

$$\alpha(r) = \sum_{S \subseteq ch(v)} \prod_{w \in S} \alpha(r-1)$$

so that

$$\alpha(r) = \sum_{j=0}^{d} \binom{d}{j} \alpha(r-1)^j \times 1^{d-j} = (1+\alpha(r-1))^d$$

by the Binomial Theorem.

Noting that $1 \leq \alpha(r-1)/\alpha(0)$ for every $r \geq 1$ we obtain,

$$\alpha(r) \leq \left(\frac{1+\alpha(0)}{\alpha(0)}\right)^d \times \alpha(r-1)^d$$

which reduces to

$$\alpha(r) \leq \left(\frac{1+\alpha(0)}{\alpha(0)}\right)^{\sum_{j=1}^{l} d^j} \times \alpha(r-l)^{d^l}$$

valid while $r - l \geq 1$, so that

$$\alpha(r) \quad \leq \quad \left(\frac{1+\alpha(0)}{\alpha(0)}\right)^{\sum_{j=1}^{r-1} \; d^j} \times (1+\alpha(0))^{d^r}$$

Simplifying to

$$\alpha(r) \quad \leq \quad \left(\frac{1+\alpha(0)}{\alpha(0)}\right)^{\frac{d^r-1}{d-1}-1} \times (1+\alpha(0))^{d^r}$$

Recalling that $\alpha(0) = (k-t)/(t(d-1)+1)$ and $d^r = t(d-1)+1$

$$\alpha(r) \quad \leq \quad \left(\frac{t(d-2)+k+1}{k-t}\right)^{t-1} \times \left(\frac{t(d-2)+k+1}{t(d-1)+1}\right)^{t(d-1)+1}$$

so that

$$e(k,t,d) \;\leq\; \frac{(t(d-2)+k+1)^{td}}{(k-t)^{t-1} \times (t(d-1)+1)^{t(d-1)+1}}$$

as claimed. □

Some specific consequences of Thm. 4 are given in

**Corollary 3** *For $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$ whose value graph is a strict tree $\langle W_v, F_v \rangle$ and $x$ such that $\eta(x) = v$,*

a. *The decision problems* SBA$(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle, x)$ *and* OBA$(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle, x)$ *are polynomial time decidable when $\langle W_v, F_v \rangle$ has $t$ vertices with more than 1 child, no vertex with more than $d$ children and $d \times t$ is constant, i.e. independent of $k = \mathcal{V}$.*
b. *If $\langle W_v, F_v \rangle$ has exactly one vertex ($v$) with more than one child and $|ch(v)| = d$ then* SBA$(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle, x)$ *and* OBA$(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle, x)$ *are decidable by an algorithm taking at most*

$$\left(\frac{k+d-1}{d}\right)^d i \times \; O(|\mathcal{X}|) \quad \textit{steps}$$

**Proof:** Both cases are immediate from the general upper bound given in Thm. 4(b). □

## 3. Beyond trees and chains

The fact that, in some circumstances, the structure of the value graph yields polynomial time solutions for SBA and OBA, as demonstrated by Corollary 3(a) and (b) (when $d$ is constant) motivates considering forms other than the trees and chains considered in Section 2. We note that our analysis of the preceding section easily extends to value graphs which are *acyclic*.[4]

[4] Value graphs which are trees in the sense used earlier may contain cycles, e.g. when the underlying VAF contains arguments $\{x, y, z\}$ with $\eta(x) = \eta(y)$, and $\langle x, z \rangle$, $\langle z, y \rangle \in \mathcal{A}$.

A natural development would be to find analogous construction for "tree-like" structures, i.e. value graphs with *bounded treewidth*, see e.g [5,1]. We recall that a *tree decomposition* of a graph $G(V,E)$[5] is a pair $\langle X, \langle I, F\rangle\rangle$ where $\langle I, F\rangle$ is a tree and $X$ is a collection $\{X_i \;:\; i \in I\}$ of $|I|$ subsets of $V$ for which $V = \cup_{i\in\; I}\; X_i$; for every edge $\langle v, w\rangle \in E$ there is at least one $i \in I$ for which $\{v, w\} \subseteq X_i$; for every $i$, $j$, $k \in I$ should $j$ occur on the (unique) path from $i$ to $k$ in $\langle I, F\rangle$ then $X_i \cap X_k \subseteq X_k$. The *width* of a tree decomposition $\langle X, \langle I, F\rangle\rangle$ is $\max_{i\in I}\; |X_i| - 1$ and the *treewidth* of $G(V,E)$, denoted $tw(G)$, is the minimum width over all tree decompositions of $G(V,E)$.

For many problems that are computationally hard in general, polynomial time algorithms exist when instances are restricted to those whose treewidth is constant. Given the results of Section 2, which can be interepreted as dealing with value graphs whose treewidth is 1, it seems reasonable to look for related constructions for value graphs whose treewidth is constant (but greater than 1). Noting that the basic component of a tree decomposition is the structure $\langle I, F\rangle$ it is plausible that further limiting the structure of $\langle I, F\rangle$, e.g. to chains, will yield efficient methods for those cases not covered by Corollary 3(a). In fact such expectations turn out to be over optimistic.

**Theorem 5**

a. *The decision problem* SBA *is* NP*–complete even when restricted to instances* $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta\rangle$ *whose value graph* $\langle \mathcal{V}, \mathcal{B}\rangle$ *has* $tw(\langle \mathcal{V}, \mathcal{B}\rangle) = 2$ and *a witnessing tree decomposition* $\langle \mathcal{I}, \mathcal{F}\rangle$ *which is a chain.*
b. *The decision problem* OBA *is* co*NP–complete even when restricted to instances* $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta\rangle$ *whose value graph* $\langle \mathcal{V}, \mathcal{B}\rangle$ *has* $tw(\langle \mathcal{V}, \mathcal{B}\rangle) = 2$ and *the tree decomposition* $\langle \mathcal{I}, \mathcal{F}\rangle$ *witnessing this is a chain.*

**Proof:** Both parts use the reductions given in [9] in order to classify these problems in VAFs when the supporting Dung-style AF, $\langle \mathcal{X}, \mathcal{A}\rangle$ is a tree.

Consider the VAF, illustrated for the case

$$\varphi = (z_1 \vee z_2 \vee z_3)(\neg z_2 \vee \neg z_3 \vee \neg z_4)(\neg z_1 \vee z_2 \vee z_4)$$

in Fig. 4, used in this reduction from 3-SAT to SBA. The value graph $\mathcal{G}(\mathcal{V}, \mathcal{B})$ of this VAF has $\mathcal{V} = \{p_i, n_i \;:\; 1 \leq i \leq n\} \cup \{c\}$, where $n$ is the number of propositional variables in the instance of 3-SAT, and $\mathcal{B}$ contains

$$\{\langle p_i, n_i\rangle,\ \langle n_i, p_i\rangle,\ \langle p_i, c\rangle,\ \langle n_i, c\rangle \;:\; 1 \leq i \leq n\}$$

This graph has a width 2 tree decomposition whose structure is a chain, i.e. the decomposition of Fig. 5. It follows that deciding SBA when value graphs have treewidth 2 defining chain structures is as hard as deciding SBA in general. □

[5] For reasons that are clear from the definition we do not need to distinguish directed and undirected cases.

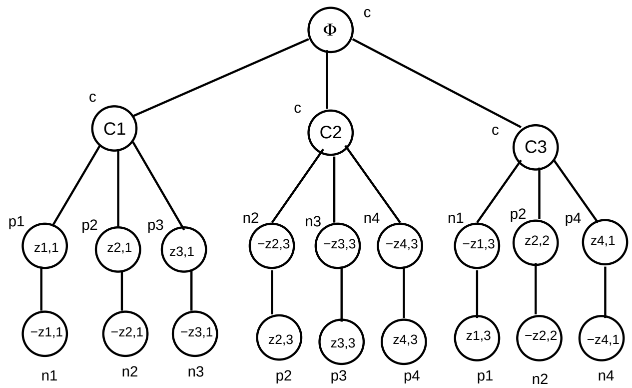


**Figure 4.** The tree using in reducing SBA from [9]

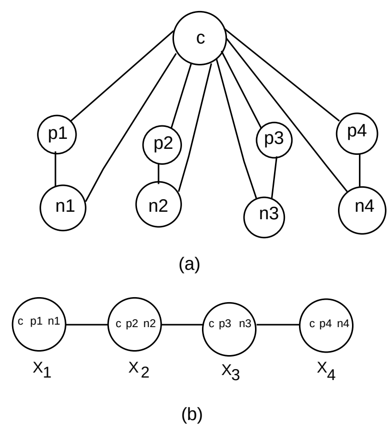


**Figure 5.** The value graph of Fig 4 and its width 2 tree decomposition

## 4. Conclusions

This paper has considered the question of identifying tractable special cases for the decision problems Subjective and Objective Acceptance (SBA, OBA) in value-based argumentation. In contrast to acceptabilty concepts in standard Dung style argumentation frameworks, only the case of symmetric AFs had been known to lead to polynomial time approaches for these problems. By considering properties of the so-called *value graph* – the directed graph structure defined by considering the *values* involved in conflicting arguments – we have identified an extensive, further, class of systems for which SBA and OBA admit polynomial time solutions: specifically those whose value graph is a tree in which the product of $t$ – the number of vertices with more than one child – and $d$ – the maximum number of children of any vertex in the tree – is constant, i.e. indepedent of the number of values. Unfortunately, and providing a further indication of the extent to which value-based argumentation has proven resistant to tractable solution methods, attempts to extend these ideas from value graphs which are trees, equivalent to the class of graphs whose *treewidth* equals 1, to value graphs with bounded treewidth encounter difficulties: even if the value graph has treewidth 2 and the structure of the witnessing tree decomposition is a chain, SBA and OBA remain NP–complete, resp. coNP–complete. Nevertheless despite the failure of bounded treewidth to yield effective solutions in general, given that some progress can be made with restricted structures on the value graph, a natural development would be to consider other graph-theoretic restrictions: a possible candidate structure, and the subject of current investigation is the class of *bipartite* value graphs. For these the reduction of Thm. 5 is inapplicable: the value graph in this instance being tripartite.

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